SECTION B

NUMERICAL METHODS 1 (3.09) taught by David Ham

Candidates being examined in "Numerical Methods 1" should answer at least one question from section B.

B1. (i) Convert the numbers in the following problems into 4 bit two's compliment signed binary integers before performing the calculation and converting back to base 10:

(a) $5 + 2$

Answer:

$$\begin{aligned} &0101_2 \\ +&0010_2 \\ =&0111_2 \\ =&7_{10} \end{aligned}$$

2 points for the conversions, 2 points for the sum.

(4 marks)

(b) -2×2

Answer:

$$\begin{aligned} &1110_2 \\ \times&0010_2 \\ =&\not{1}1100_2 \\ =&1010_2 \\ =&-4 \end{aligned}$$

2 points for the conversions, 3 points for the product including properly dropping the overflow.

(5 marks)

(ii) For each of the following Python functions, write a mathematical expression using only matrices and vectors which performs the same operation. In each case, state whether each input and output is a matrix or vector.

(a)
```
def function_1(a,b):
    from numpy import dot, zeros

    c=zeros((a.shape[0],b.shape[1]))

    for i in range(a.shape[0]):
        for j in range(b.shape[1]):
            c[i,j]=dot(a[i,:],b[:,j])

    return c
```

Answer:

$$\mathrm{C} = \mathrm{AB} \tag{1}$$

where A, B and C are all matrices.

2 marks for the operation, 2 marks for the types

(4 marks)

(b)
```python
def function_2(a,b):
    from numpy import dot, zeros

    c=zeros(a.shape[1])

    for i in range(a.shape[1]):
        c[i]=dot(a[:,i],b)

    return c
```

Answer:

$$\mathbf{c} = \mathrm{A}^{\mathrm{T}}\mathbf{b} \tag{2}$$

or equivalently:

$$\mathbf{c} = \mathbf{b}\mathrm{A} \tag{3}$$

where A is a matrix and **b** and **c** are vectors.
2 marks for the operation, 2 marks for the types

(4 marks)

(iii) The forward difference approximation to the derivative of a function f at a point x is given by:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h^p)$$

for some p. By expanding the first two terms of the Taylor series for f, derive this formula and find the value of p.

Answer: The Taylor series expanded to two terms is:

$$f(x+h) = f(x) + hf'(x) + \mathcal{O}(h^2)$$

(3 marks)

By rearranging, we get:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{\mathcal{O}(h^2)}{h}$$

(3 marks)

Knowing the rule for dividing h through $\mathcal{O}$ gives:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$$

and therefore $p = 1$ *(2 marks)*

(8 marks)

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B2. (i) Consider the number -2.5

(a) Write the number in the form:

$$s1.m \times 2^{e-b} \tag{4}$$

Use a floating point format with 3 exponent bits and 4 mantissa bits, and a bias of 3. All the numbers must be written in binary.

Answer: 2.5 is 10.1_2 or $1.01_2 \times 2^1$. We need $e - 3 = 1$ so $e = 4 = 100_2$. The sign is negative so in the prescribed form it's

$$11.01_2 \times 2^{100_2 - 11_2}$$

1 mark for sign, 2 each for exponent and mantissa and 1 for correctly applying the bias.

(6 marks)

(b) Convert this number into the bit pattern which would actually be stored, according to the layout provided on the formula sheet.

Answer: 11000100

If the candidate makes a mistake on the first part but the bit pattern is consistent with their answer, they get the marks.

(2 marks)

(ii) Consider the following piece of Python code:

```
def find_root(f, a, b, eps):
    if f(a)*f(b) > 0:
        return None, 0

    while b-a > eps:
        m = (a + b)/2.0
        print a,b,b-a,m
        if f(a)*f(m) <= 0:
            b = m  # root is in left half of [a,b]
            print 'Root in left half'
        else:
            a = m  # root is in right half of [a,b]
            print 'Root in right half'
    print m
    return m
```

(a) Which of the root finding algorithms which we studied in the course does this function implement?

Answer: The bisection method. This should be obvious because of the interval halving step and the fact that there is no gradient calculation involved.

(2 marks)

(b) Imagine I run the following Python code:

```
def g(x):
    return x

find_root(g, -0.5, 1., 0.5)
```

What is printed out?

Answer:

```
 -0.5 1.0 1.5 0.25
Root in left half
-0.5 0.25 0.75 -0.125
Root in right half
-0.125
```

3 marks for calculating correct values, 2 marks for getting the roots in the right halves. 1 mark for terminating at the right point. 1 mark for the correct root.

(7 marks)

(iii) Suppose we have an $n \times n$ matrix A and three n-vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{b} \neq \mathbf{c}$. Suppose also that:

$$\mathrm{A}\mathbf{b} = \mathbf{d}$$
$$\mathrm{A}\mathbf{c} = \mathbf{d}$$

(a) Show that there must be some non-zero vector $\mathbf{e}$ such that:

$$\mathrm{A}\mathbf{e} = \mathbf{0}$$

Answer: Subtracting the two equations gives:

$$\mathrm{A}\mathbf{b} - \mathrm{A}\mathbf{c} = \mathbf{0} \tag{5}$$

(2 marks)

Using the distributive law for matrices:

$$\mathrm{A}(\mathbf{b} - \mathbf{c}) = \mathbf{0} \tag{6}$$

(2 marks)

We can write $\mathbf{e} = \mathbf{b} - \mathbf{c}$. Since $\mathbf{b} \neq \mathbf{c}$, $\mathbf{e} \neq \mathbf{0}$ and we are done. *(2 marks)*

(6 marks)

(b) Using the properties of linear independence, explain why the result of part B2(iii)(a) shows that the columns of A are *linearly dependent*.

Answer: Linear independence property 5 on the formula sheet states that if the columns of A are linearly independent, $\mathrm{A}\mathbf{e} = \mathbf{0}$ only if $\mathbf{e} = \mathbf{0}$. Since there is a non-zero $\mathbf{e}$ for which this is the case, the columns of A must be linearly dependent. *(2 marks)*

(2 marks)

END OF SECTION B