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# Minimizing Downside Risk in Axioma Portfolio with Options

## 1 Introduction

The market downturn in 2008 is considered by many economists to be the worst financial crisis since the Great Depression of the 1930s. During this time period, the Dow Jones Industrial Average fell 33.8%, the S&P 500 index fell 38.6%, and the NASDAQ fell 40.5%. Options are often used by portfolio managers to insure a portfolio against adverse market movements as in 2008. Many portfolio managers familiar with mean-variance optimization incorporate options within this framework, to determine the number of options to be purchased in order to minimize some measure of the portfolio risk. The most popular attempt is the delta-approximation approach with the option modeled as a composite asset, where an investment in one unit of the composite is equivalent to the delta units in the underlying asset. This amounts to the common practice of delta-hedging, where the option holdings are chosen to make the overall portfolio delta-neutral. Delta-hedging strategies can be effective, however, the portfolio must frequently be adjusted to account for the changing delta. Moreover, these delta-hedging approaches minimize portfolio movements thereby limiting both the downside as well as the upside of the portfolio. While delta-hedging strategies are common, a simple delta-hedging model in a mean-variance framework does not give an option portfolio that provides good downside protection. The primary reason is that this model fails to capture the limited downside risk involved in purchasing the option, where the most one can lose is the option premium however far the market goes against you.

Sivaramakrishnan et al. (May 2011) present a new methodology to minimize the downside risk of a portfolio containing equities, indices, and options. This methodology is now available in Axioma Portfolio™. Given a portfolio containing equities and indices to be protected from large downside moves, a sell-side firm or bank can use our methodology to determine the options to be purchased in order to minimize the downside risk of the overall portfolio. The options are American or European calls or puts on the individual equities and indices in the portfolio. They can have different strikes, volatility, dividends, and time to expiration. The major feature of the new methodology is a downside risk measure called Worst-Case-Value-At-Risk (WCVaR) for portfolios containing equities, indices, and long positions on options. The WCVaR risk measure assumes that the equity returns are from an elliptic distribution that includes some fat-tailed distributions. In the WCVaR risk measure model, these returns lie in an ellipsoidal uncertainty set that is determined from the mean and the covariance of the equity returns. The WCVaR risk measure value is the worst (maximum) portfolio loss that is obtained when the equity returns are unknown and can take any possible value in the uncertainty set. This measure was first introduced for equity-only portfolios in El Ghaoui et al. (2003) and later extended to portfolios containing options in Zymler et al. (2011). The option returns in our WCVaR measure are based on a convex function of the underlying asset returns. This function is calculated from the Black-Scholes pricing function for an European option and from the Cox-Ross-Rubinstein binomial asset pricing function for an American option. An important feature of the WCVaR risk measure is that it can be efficiently computed and optimized for a large portfolio containing options by solving a convex second-order cone program. This is done by using the powerful interior-point optimizer in Axioma Portfolio.

We present excerpts from Sivaramakrishnan et al. (May 2011) in this article in order to highlight the important facets of this new methodology in Axioma Portfolio.

# 2 The WCVaR downside risk measure

Consider a portfolio with $n$ equities with holdings $w$ and returns $r$. Let $\mathcal{E}$ denote the equity set. The equity returns are random variables with mean $\mu$ and covariance $Q$. Let $r^p = r^T w$ denote the portfolio return. The portfolio also contains $m$ options on the individual equities. The options are American and European call and put options with conceivably different strike prices, expiration dates, volatilities, and risk-free rates. Let $\mathcal{O}$ denote the set of options, $r^o$ denote the option returns, and $w^o$ denote the option holdings. Options cannot be shorted, i.e., their holdings have to be non-negative. Let $u_j \in \mathcal{E}$ denote the underlying equity for option $j$. The option returns are functions of the underlying equity returns and the return for the $j$th option is given by

$$r_j^o \quad = \quad \max\{-1, f(r_{u_j})\}$$

**(1)**

where $f(r_{u_j})$ is a convex function of the underlying return. In Axioma Portfolio, $f(r_{u_j})$ is designed to be a convex approximation to the Black-Scholes pricing function for European options and the Cox-Ross-Rubinstein binomial tree pricing function for American options.

For a given set of asset holdings $w$ and option holdings $w^o$; consider the Worst-Case-Value-at-Risk (WCVaR) risk measure

$$\begin{array}{rcl} \mathrm{WCVaR}(w,w^o) & = & -\min\limits_{r,r^o} \quad w^T r + (w^o)^T r^o \\ & & \text{s.t.} \quad ||Q^{-1/2}(r-\mu)|| \;\leq\; \kappa, \\ & & \qquad\qquad\qquad r_i \;\geq\; -1, \;\; i \in \mathcal{E}, \\ & & \qquad\qquad\qquad r^o_j \;\geq\; \max\{f_j(r_{u_j}), -1\}, \;\; j \in \mathcal{O} \end{array}$$

**(2)**

in Axioma Portfolio, where $\kappa > 0$ is a parameter that determines the size of the ellipsoidal uncertainty set

$$\mathcal{U} \;=\; \left\{ r \in I\!R^n : ||Q^{-1/2}(r-\mu)|| \leq \kappa, \;\; r_i \geq -1, \;\; i \in \mathcal{E} \right\}$$

**(3)**

containing the equity returns. The WCVaR risk measure value is the worst (maximum) portfolio loss that is obtained when the equity returns are unknown and can take any possible value in the uncertainty set $\mathcal{U}$. This way, the WCVaR risk measure gives a fairly conservative value for the total loss experienced by a portfolio within a certain time horizon. The measure is more conservative for larger values of $\kappa$.

The WCVaR risk measure in (2) is a convex risk measure only under the restriction that we go long on options. In Axioma Portfolio, WCVaR also handles indices and options on indices; in particular, an index is treated as a composite asset that holds the index equities in proportion to their holdings in the index. It is shown in Sivaramakrishnan et al. (May 2011) that WCVaR exhibits the same trend as other commonly used downside risk measures such as Value-at-Risk (VaR) and Conditional-Value-At-Risk (CVaR) for portfolios containing different levels of option holdings. WCVaR has the following advantages over VaR and CVaR:

1. WCVaR can be computed and optimized efficiently using the powerful interior point optimizer in Axioma Portfolio. In contrast, VaR and CVaR are commonly computed and optimized using Monte-Carlo sample based approaches and the solution times grow quickly with the number of samples.
2. WCVaR is a coherent risk measure. As a result, the WCVaR of a portfolio cannot exceed the weighted sum of the WCVaRs of the constituent assets in the portfolio. This way,

WCVaR encourages diversification of assets in a portfolio as favored by financial practitioners. It has been shown that VaR is not a coherent risk measure.

# 3 Controlling downside risk with new option risk measure

The WCVaR risk measure is used to control the downside risk of a portfolio containing equities, indices, and options. In Axioma Portfolio, this can be done in the following ways:

1. Minimizing the WCVaR portfolio risk measure subject to: (a) option budget, and (b) go long on options. This gives the optimization problem

$$\begin{array}{lrcl} \min_{w^o} & \mathrm{WCVaR}(w, w^o) & & \\ \text{s.t.} & \sum_{j \in \mathcal{O}} w_j^o & \leq & \mathrm{OB}, \\ & w_j^o & \geq & 0, \ j \in \mathcal{O} \end{array}$$

**(4)**

where $\mathrm{OB} > 0$ is the option budget.

2. Minimizing the option investment subject to: (a) WCVaR risk measure be below a certain threshold, and (b) go long on options. This gives the optimization problem

$$\begin{array}{lrcl} \min_{w^o} & \sum_{j \in \mathcal{O}} w_j^o & & \\ \text{s.t.} & \mathrm{WCVaR}(w, w^o) & \leq & \theta, \\ & w_j^o & \geq & 0, \ j \in \mathcal{O} \end{array}$$

**(5)**

where $\theta > 0$ is a certain risk threshold.

It must be emphasized that Axioma Portfolio also allows other objectives and constraints on the option variables.

# 4 The delta-hedging strategy

There have been attempts to incorporate options in a mean-variance optimization framework. The most popular attempt is to model an option as a composite asset where an investment in one unit of the composite is equivalent to delta units in the underlying asset. The delta of an option is defined as the rate of change of the option price with respect to the change in the price of the underlying asset.

We will now describe an optimization model for the delta-hedging strategy. The delta-hedging strategy basically models the return for the $j$th option as

$$r_j^o \;=\; \Delta_j \left(\frac{S_{u_j}}{c_j}\right) r_{u_j}$$

**(6)**

where $\Delta_j$is the option delta; $c_j$is its price; $u_j$is the underlying asset, $S_{u_j}$is its share price, and $r_{u_j}$is its return. The delta-hedging model fails to capture the limited downside risk involved in purchasing an option, where the most one can lose is the option premium however far the market goes against you. It is shown in Sivaramakrishnan et al. (May 2011) that delta-hedging minimizes the following risk measure

$$\mathrm{DHVaR}(w, w^o) \;=\; \theta ||Q^{1/2}(w + Dw^o)|| - \mu^T(w + Dw^o)$$

**(7)**

where $\theta > 0$controls the weight assigned to the risk term in $\mathrm{DHVaR}(w, w^o)$, $\mu$and $Q$are the mean the covariance of the equity returns, and $D$is the equity-option loading matrix. The $j$th column of the $D$matrix has one non-zero entry $\Delta_j \left(\frac{S_k}{c_j}\right)$in the $k$th position. It is also shown in Sivaramakrishnan et al. (May 2011) that minimizing the delta-hedging risk measure (7) is closely related to the strategy of choosing the option holdings so as to make the overall portfolio delta-neutral. In a delta-neutral portfolio, the delta of each equity is offset by the delta of the option positions on it. For an asset with several options, there are many ways to choose the option holdings so that the overall portfolio is delta-neutral. To handle these multiple solutions, we choose the delta-neutral option portfolio that minimizes the option investment.

# 5 Backtest results

Our backtest is run from January 2005 to August 2010. The portfolio is hedged on the 3rd Friday of every month. Our universe is restricted to equities in the S&P500 and the SPDR S&P500 ETF. We consider one-month options on the individual equities as well as on the SPDR S&P500 ETF. The option data for each month is taken from the *OptionMetrics* database. For each time period, we construct a long-only equity portfolio that closely tracks the S&P500, where we limit the number of equities held to 100 and do not hold the ETF. The equity holdings satisfy a budget constraint. The equity budget for the succeeding time periods is chosen by rolling forward the equity and option holdings from the previous period. We refer the reader to Sivaramakrishnan et al. (May 2011) for more details about the setup.

We would like to emphasize that we are not interested in how this equity portfolio is generated in this report. We assume that a sell-side firm is given this equity portfolio from a portfolio manager. This firm uses our WCVaR hedging methodology to determine the options to be purchased in order to minimize the downside risk of the overall portfolio. We assume that the hedging time horizon matches the portfolio rebalancing time horizon in this report. Sivaramakrishnan et al. (May 2011) present results with a daily hedging approach, where the equity portfolio is rebalanced by a portfolio manager on the 3rd Friday of every month and the portfolio is hedged on every trading day between any two portfolio rebalancings.

We compare the following option hedging strategies:

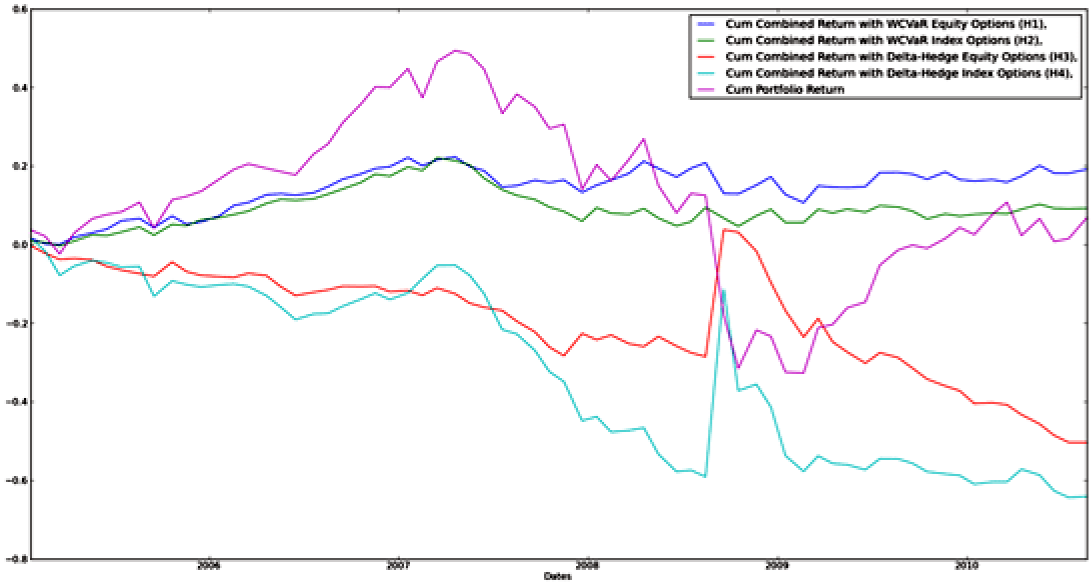
**H1:** WCVaR option measure with options on the individual equities only.

**H2:** WCVaR option measure with options on the SDPR S&P500 ETF only.

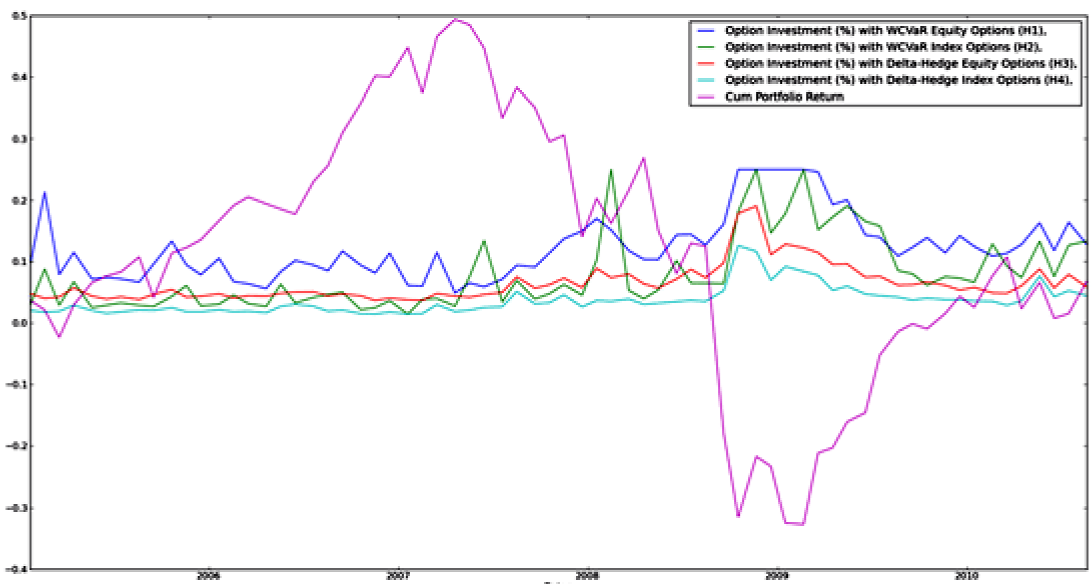
**H3:** Delta-hedging option measure with options on the individual equities only.

**H4:** Delta-hedging option measure with options on the SDPR S&P500 ETF only.

All four hedging strategies are long on options. The WCVaR hedging strategies H1 and H2 solve the optimization problem (2) to determine the option holdings in each time period. The delta-hedging strategies H3 and H4 choose the delta-neutral option portfolio that minimizes the option investment. For the WCVaR hedging strategies H1 and H2, we choose $\kappa = 1.0$. For the delta hedging strategies H3 and H4, we choose $\theta = 1.0$ in (7). For all hedging strategies, the option budget is chosen to be 25% of the equity budget for that period. The expected return $\mu$ is chosen to be the risk-free rate for all the equities over the one month hedging time horizon. The asset covariance matrix $Q$ is taken from the US2AxiomaMH risk model and these values are scaled to match the one month hedging time horizon.




(a) Cumulative portfolio returns for hedging strategies
[(Click for a larger view.)]




(b) Option Investment for hedging strategies
[(Click for a larger view.)]

**Figure 1**: Backtest results with the four hedging strategies

Panel (a) of Figure 1 compares the cumulative portfolio returns for the four hedging strategies with the cumulative return of the equity-only portfolio. The cumulative portfolio returns for the WCVaR option hedging strategies H1 and H2 are better than that of the equity-only portfolio over the same period. On the other hand, the delta-hedging strategies H3 and H4 lose money; the cumulative portfolio returns for these strategies are much worse than that of the equity-only portfolio over the same period. Examining the option investment for the four hedging strategies from Panel (b) of Figure 1, we find that the WCVaR option hedging strategies use the entire available budget, especially in late 2008 and early 2009 during the financial crisis. The delta-hedging strategies, however, only use between 5 - 15% of the equity budget during this period.

To better compare the WCVaR hedging strategy with the delta-hedging strategy, we consider the index options purchased by the strategies (H2 and H4) on the 3rd Friday of October 2008, when there was turmoil in the financial markets with the collapse of Lehman. Table 1 presents information on the options purchased by these hedging strategies. We assume that these options are held throughout the hedging time horizon and sold on their expiration date. The last column in Table 1 contains the option payoffs for different index prices $S$ on the 3rd Friday of November 2008. The SDPR S&P 500 index was trading at $93.21 per share on the 3rd Friday of October 2008. It was trading at $79.52 a share on the 3rd Friday of November 2008, that represents a 15% drop in its share price. So, by exercising the WCVaR index option at expiration, a portfolio manager was able to add $15.51 to the portfolio value. On the other hand, the option purchased by the delta-hedging strategy (H4) expired worthless, and the combined portfolio lost $11.16 due to this option. The delta-hedging strategy (H4) never makes more money than the WCVaR hedging strategy (H2) with these purchased options; it loses less money only if the SPDR S&P500 index was trading at or more than $110 a share on the 3rd Friday of November 2008. In this case, the portfolio loses $19.36 in options for the WCVaR hedging strategy (H2), and loses $11.16 in options for the delta-hedging strategy (H4).

| Hedging Strategy | Shares | Strike ($) | Cost ($) | Payoff ($) |
|---|---|---|---|---|
| WCVaR (H2) | 1.14 | 110 | 16.93 | $1.14\,(\max\{110 - S,\ \ 0\} - 16.93)$ |
| Delta (H4) | 32.35 | 56 | 0.35 | $32.35\,(\max\{56 - S,\ \ 0\} - 0.35)$ |

**Table 1**: Index options purchased by hedging strategies in October 2008

Figure 2 compares the probability distribution function (pdf) plots for the equity-only portfolio return $r^p = w^T r$ and the combined portfolio returns $r^c = w^T r + (w^o)^T r^o$ for the four hedging strategies in October 2008, when the equity returns are sampled from a normal distribution with mean $\mu$ and covariance $Q$, where $\mu$ is the risk-free rate and $Q$ is taken from the US2AxiomaMH risk model.

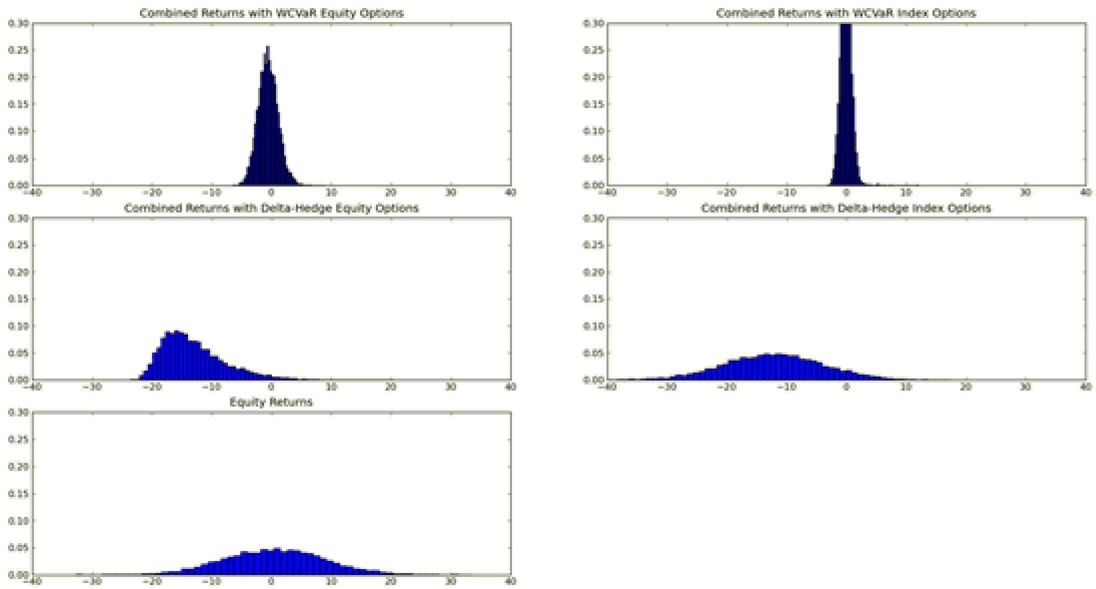



**Figure 2**: Monte-Carlo results with the four hedging strategies
(Click for a larger view.)

The pdf plots for $r^c$computed from the WCVaR hedging strategies have significantly shorter left tails than the pdf plots for $r^p$. The WCVaR index option hedging strategy provides the best downside risk protection, since it provides the shortest left tail for $r^c$. On the other hand, the pdf plots for $r^c$computed from the delta-hedging strategies have a significant mass in the lower tail of the return distribution. So, it appears that the WCVaR index option strategy (H2) is the best overall strategy and provides the best downside protection. The WCVaR index option strategy also performs better in other statistics that measure the downside risk of the portfolio, and we refer the reader to Sivaramakrishnan et al. (May 2011) for further details.

# 6 Conclusions

Options are commonly used by traders to provide insurance to equity portfolios. Axioma Portfolio now has a new methodology to minimize the downside risk of a portfolio containing stocks, indices, ETFs, and long positions in American and European call and put options on the underlying assets.

1. The option holdings to minimize the downside risk of an equity portfolio are obtained by optimizing a Worst-Case-Value-at-Risk (WCVaR) risk measure. This measure can handle American and European options with different time to expiration. The return for each option is a convex function of the underlying asset return, that uses the Cox-Ross-Rubinstein pricing function for an American option and the Black-Scholes pricing function for a European option. The WCVaR risk measure has the following advantages: (a) it can be calculated and optimized efficiently, (b) it is a coherent risk measure that encourages diversification of assets in a portfolio.
2. Our backtest results indicate that the WCVaR based hedging approach provides better downside protection than the delta-hedging model that is commonly used to model options in a mean-variance optimization framework. The WCVaR based hedging strategy

is able to capture the asymmetric return distribution of a portfolio containing options, since it models an option return as a convex approximation of the underlying asset return that is derived from the option pricing formula. This feature is absent in the linear delta-hedging option model, that fails to capture the limited downside risk in purchasing an option.

3. Among the WCVaR hedging strategies, the index option strategy provides better downside protection than the equity option strategy. We believe this is because: (a) Options on the index are much cheaper than options on the constituent equities, and (b) Options on an index are more liquid, and so they are priced more efficiently. Hence it is better to hedge an equity portfolio by using the WCVaR risk measure with options on an index with a high correlation to the equity portfolio.

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