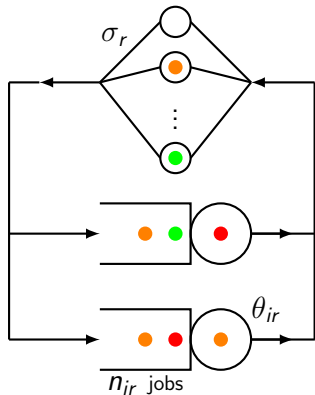


Accelerating Performance Inference over Closed Systems by Asymptotic Methods

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Closed queueing networks



K nodes

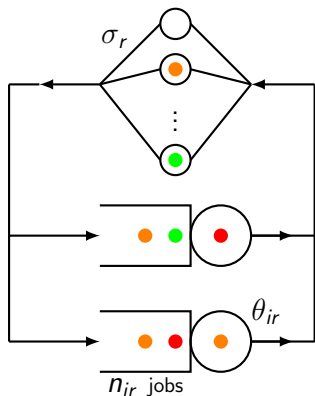
R classes

N jobs, N_r in class r

Product-form ($\sigma_r = 0$):

$$P(\mathbf{n}) = \frac{1}{G} \prod_{i=1}^K n_i! \prod_{r=1}^R \frac{\theta_{ir}^{n_{ir}}}{n_{ir}!}$$

Closed queueing networks



K nodes

R classes

N jobs, N_r in class r

Product-form ($\sigma_r = 0$):

$$G = \sum_{\mathbf{n}} \prod_{i=1}^K n_i! \prod_{r=1}^R \frac{\theta_{ir}^{n_{ir}}}{n_{ir}!}$$

See paper for case $\sigma_r > 0$.

Motivating example

- ▶ Problem: **service demand inference**
 - ▶ Q_{ir} : empirical mean queue-length
- ▶ Maximum-likelihood estimation (MLE)

$$\max_{\theta \in \Omega} \sum_{i,r} Q_{ir} \log \theta_{ir} - \log G$$

where $\theta \equiv (\theta_{ir})$ and $G \equiv G(\theta)$.

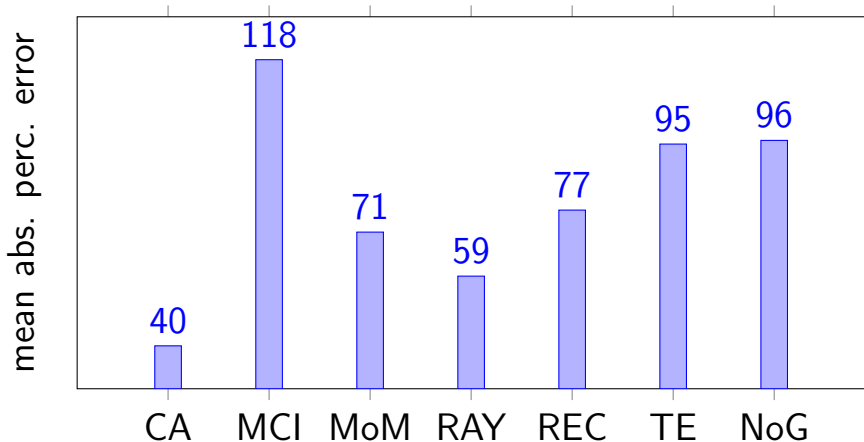
Related work

Computing the normalizing constant G :

- ▶ **CA**: Convolution algorithm
 - ▶ **RECAL**: Recursion by chain
 - ▶ **MoM**: Method of moments
 - ▶ **RAY**: Ray method
 - ▶ **MCI**: Monte Carlo integration
 - ▶ **TE**: Taylor expansion
 - ▶ ...
- } exact methods

Motivating example

- ▶ 60s run, 3 nodes, 3 classes, 120 jobs



This prompts us to revisit G 's computational theory.

Interpolation theory

- ▶ Consider $f(x)$ and data points $x = \theta_1, \dots, \theta_K$
- ▶ Newton's interpolation polynomial

$$N(x) = \prod_{i=1}^K (x - \theta_i) \cdot \underbrace{[\theta_1, \dots, \theta_K] f(x)}_{\text{divided difference}} + \dots$$

- ▶ Explicit form for divided differences

$$[\theta_1, \dots, \theta_K] f(x) = \sum_{k=1}^K \frac{f(\theta_k)}{\prod_{i \neq k} (\theta_k - \theta_i)}$$

Single-class models

- ▶ Explicit form for single-class models ($R = 1$)

$$G = \sum_{\mathbf{n}} \prod_{k=1}^K \theta_k^{n_k} = \sum_{k=1}^K \frac{\theta_k^{N+K-1}}{\prod_{i \neq k} (\theta_k - \theta_i)}$$

- ▶ Matching the definitions

$$G = [\theta_1, \dots, \theta_K] x^{N+K-1}$$

A new integral form

- ▶ **Hermite-Genocchi formula** for divided differences

$$G = \frac{(N + K - 1)!}{N!} \int_{\Delta_K} (\theta_1 u_1 + \dots + \theta_K u_K)^N d\mathbf{u}$$

on the simplex $\Delta_K = \{|\mathbf{u}| = 1, \mathbf{u} \geq \mathbf{0}\}$.

- ▶ **Laplace transform** on volume of Δ_K yields the McKenna-Mitra integral form for G over \mathbb{R}_+^K .

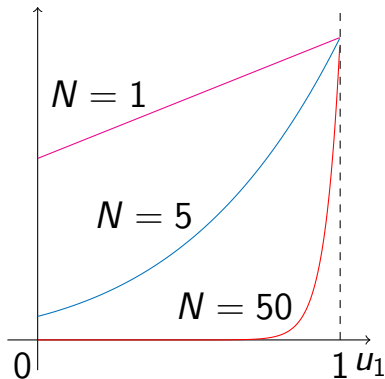
Example

$K = 2$ queues

$u_1 + u_2 = 1$

$\theta_1 > \theta_2$

$$G \propto \int_{\Delta_K} (\theta_1 u_1 + \theta_2 u_2)^N d\mathbf{u}$$



Multiclass analysis

- ▶ We generalize the result to multiclass models

$$G = \frac{(N + K - 1)!}{N_1! \cdots N_R!} \int_{\Delta_K} \prod_{r=1}^R (\theta_{1r} u_1 + \cdots + \theta_{Kr} u_K)^{N_r} d\mathbf{u}$$

- ▶ Proof uses multinomial theorem and

$$P(\mathbf{n}) \propto n_1! \cdots n_K! \propto \int_{\Delta_K} \prod_{k=1}^K u_k^{n_k} d\mathbf{u}$$

Computational methods

From the integral form, we derive:

- ▶ Explicit solutions
- ▶ Asymptotic expansion
- ▶ Cubature rule

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Explicit solutions

- ▶ Integral form for **single class** models ($R = 1$)

$$G = \frac{(N + K - 1)!}{N!} \int_{\Delta_K} (\theta_1 u_1 + \dots + \theta_K u_K)^N d\mathbf{u}$$

- ▶ Integral form for **multiclass** models ($R > 1$)

$$G = \frac{(N + K - 1)!}{N_1! \dots N_R!} \int_{\Delta_K} \prod_{r=1}^R (\theta_{1r} u_1 + \dots + \theta_{Kr} u_K)^{N_r} d\mathbf{u}$$

From products to sum

- ▶ Can we relate the two cases?

$$\prod_{r=1}^R x_r^{N_r} = \frac{1}{N!} \frac{\partial^{N_1} \dots \partial^{N_R}}{\partial t_1^{N_1} \dots \partial t_R^{N_R}} \left(\sum_r t_r x_r \right)^N$$

where $N = \sum_r N_r$.

- ▶ Multiclass integrand reduced to single class!

$O(N^R)$ Explicit form

- ▶ Replace derivatives with **finite differences**.
- ▶ Applying to the integrands

$$G = \sum_{\mathbf{t}: 0 \leq t_r \leq N_r} \underbrace{\prod_{r=1}^R \frac{(-1)^{N_r - t_r}}{N_r!} \binom{N_r}{t_r}}_{\text{finite difference}} \underbrace{\sum_{k=1}^K \frac{\theta_k^{N+M-1}}{\prod_{i \neq k} (\theta_i - \theta_k)}}_{\text{single-class solution}}$$

where $\theta_k = \sum_r t_r \theta_{kr}$.

- ▶ Same time complexity as CA, but $O(1)$ space

$O(N^{K+1})$ Explicit form

- ▶ Applying the idea to products in $P(\mathbf{n})$

$$G = \sum_{\mathbf{h} \geq \mathbf{0}: h \leq N} \frac{(-1)^{N-h}}{N_1! \cdots N_R!} \binom{N+K-1}{N-h} \prod_{r=1}^R \left(\sum_{k=1}^K h_k \theta_{kr} \right)^{N_r}$$

- ▶ Same time as RECAL, but $O(1)$ space

Computational methods

From the integral form, we derive:

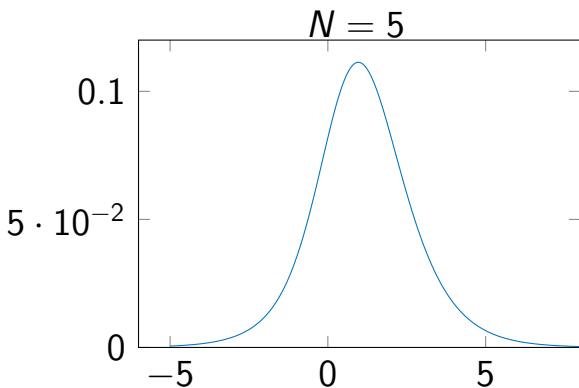
- ▶ Explicit solutions
- ▶ Asymptotic expansion
- ▶ Cubature rule

Logistic transformation

$$u_1 \rightarrow \frac{e^x}{1 + e^x}$$

$$u_2 \rightarrow \frac{1}{1 + e^x}$$

$$\left| \frac{\partial u}{\partial x} \right| = \frac{e^x}{(1 + e^x)^2}$$



- ▶ Approximate mapping to a Gaussian over \mathbb{R}^{K-1}
- ▶ Close to Gaussian also for small N

Properties

- ▶ What are the parameters of the Gaussian?
- ▶ Mean $\hat{\mathbf{u}}$ obtained iteratively

$$\hat{u}_i = \sum_{r=1}^R \frac{N_r}{N \sum_k \theta_{kr} \hat{u}_k}$$

- ▶ Asymptotically identical to mean-value analysis
- ▶ Explicit form found for the covariance matrix \mathbf{A}

Heavy-load asymptotics

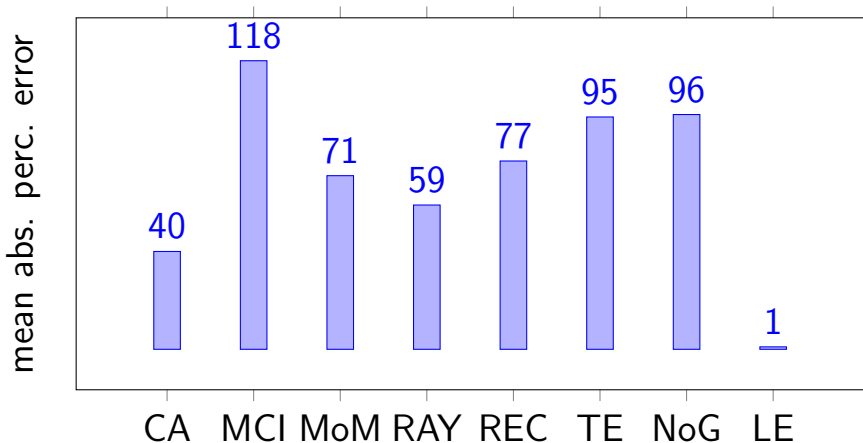
- ▶ We force ϵN self-looping jobs at each node
- ▶ For small ϵ , we get in heavy load ($N \rightarrow \infty$)

$$G \sim \frac{(N + K - 1)!}{N_1! \cdots N_R!} \sqrt{\frac{(2\pi)^{K-1}}{\det(\mathbf{A})}} \prod_{r=1}^R \left(\sum_{k=1}^K \theta_{kr} \hat{u}_k \right)^{N_r} \prod_{i=1}^K \hat{u}_i$$

- ▶ We refer to this result as **logistic expansion** (LE).

Example: MLE revisited

- ▶ 60s run, 3 nodes, 3 classes, 120 jobs

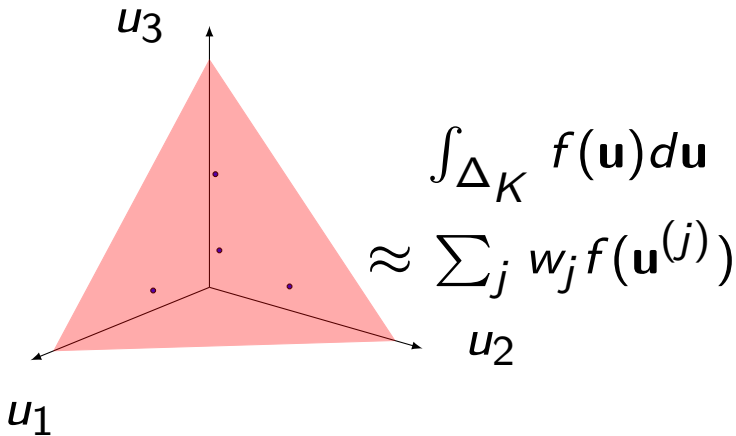


Computational methods

From the integral form, we derive:

- ▶ Explicit solutions
- ▶ Importance sampling
- ▶ Asymptotic expansion
- ▶ Cubature rule

Cubature rules



Grundmann-Möller cubature

- ▶ $O(S^K)$ samples symmetric to simplex barycenter

$$\mathbf{u}^{(j)} = \left(\frac{(2b_1^{(j)} + 1)}{(2S + K - 2s)}, \dots, \frac{(2b_K^{(j)} + 1)}{(2S + K - 2s)} \right)$$

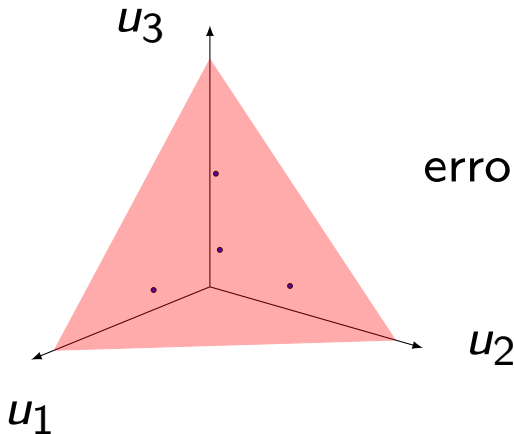
$\forall s = 0, \dots, S$ and $\mathbf{b}^{(j)} \geq \mathbf{0}$ s.t. $|\mathbf{b}^{(j)}| = S - s$.

- ▶ For G , the rule is **exact** when $S = \lceil (N - 1)/2 \rceil$
- ▶ Lower degree rules give approximations of G

Grundmann-Möller cubature

degree $S = 1$

4 points

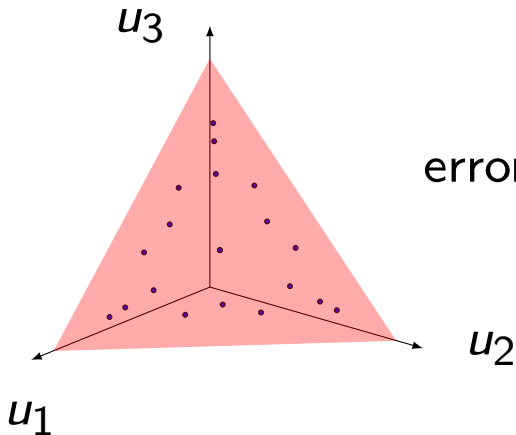


error=92.9%

Grundmann-Möller cubature

degree $S = 2$

18 points

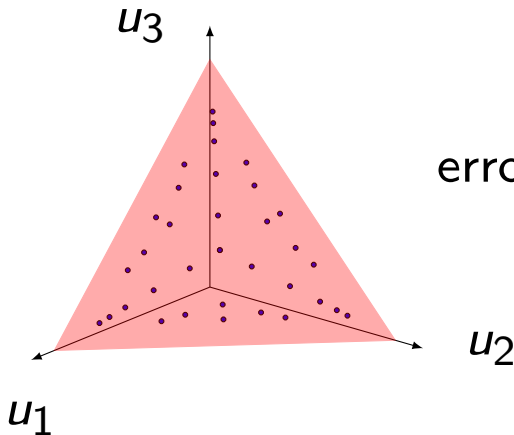


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Grundmann-Möller cubature

degree $S = 4$

35 points

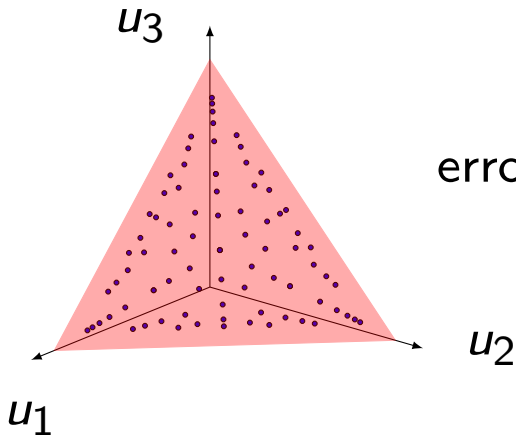


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Grundmann-Möller cubature

degree $S = 6$

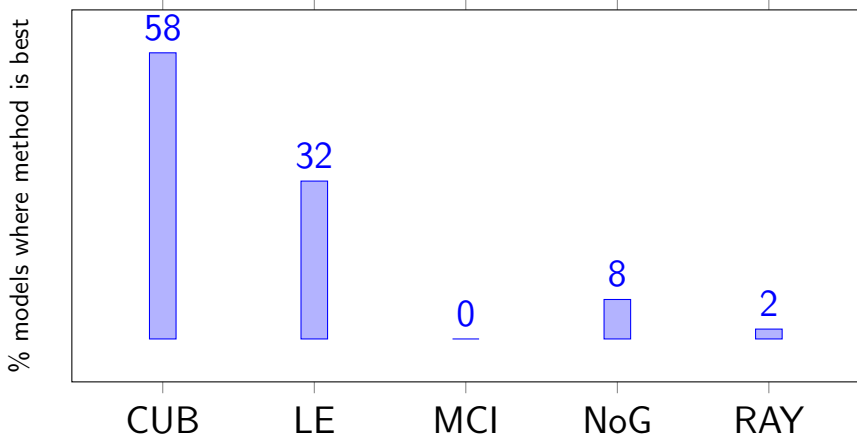
84 points



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Validation: Light-load

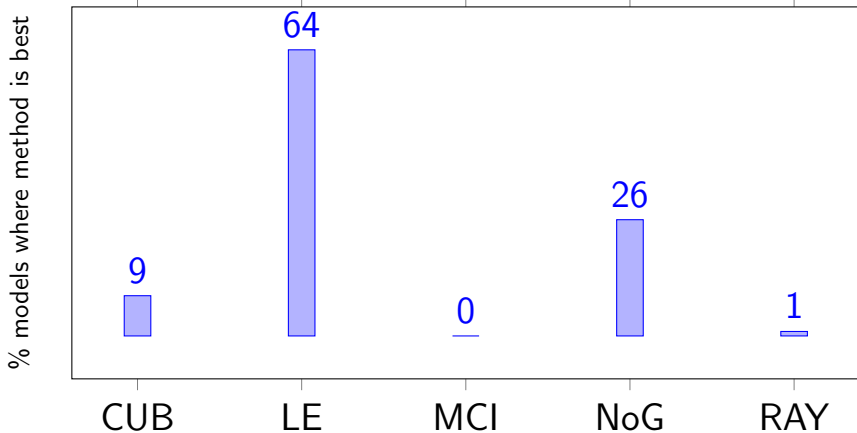
- ▶ $K, R \in [2, 32]$, $N/R = 2$, 1125 MLE problems



Cubature rule best in light load.

Validation: Heavy-load

- ▶ $K, R \in [2, 32]$, $N/R = 40$, 1125 MLE problems



Logistic expansion best in heavy load.

Conclusion

Main findings:

- ▶ Novel integral form for G on the unit simplex
- ▶ Explicit solutions
- ▶ Asymptotic expansions and cubature rule

Additional results in the paper:

- ▶ Importance sampling method
- ▶ Hybrid LE-Approximate MVA
- ▶ Models with delay nodes ($\sigma_r > 0$)
- ▶ Validation on random models